MAGNETODYNAMICS OF PLANE AND AXISYMMETRIC FLOWS OF A GAS WITH INFINITE ELECTRICAL CONDUCTIVITY

(MAGNETODINAMIKA PLOSKIKH I OSESIMMETRICHNYKH TECHENII GAZA S BESKONECHNOI ELEKTRICHESKOI PROVODIMOST'IU)

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In this paper we investigate flows of an ideal gas with infinite conductivity in a magnetic field which is parallel to the velocity of the approaching stream. It is shown that there exist two hyperbolic flow regimes, one of which occurs at subsonic velocities. In this flow regime, shock waves are inclined upstream. For certain values of the ratio between magnetic and hydrodynamic pressures, there exists an elliptic type of flow at supersonic velocities. In this regime weak shock waves do not occur, but there are strong shock waves, whose angles of inclination start from the perpendicular. We work out the simple waves for the hyperbolic regimes and construct the solutions for the problem of flow around bodies, in the linearized and second-order approximations.

1. Shock waves. Let $\mathbf{H}_1$ be the magnetic field and $\mathbf{V}_1$ the velocity vector, both parallel ahead of the shock wave, and let the shock wave form the angle σ with the direction of these vectors (Fig. 1). Then the pressure p, density ρ, velocity $\mathbf{V}$ and field $\mathbf{H}$, downstream (index 2) and upstream (index 1) of the shock wave are related by the following relations [1]:

the condition for continuity of the normal component of field

$$H_n = H_1 \sin\sigma = H_2 \sin(\sigma - \vartheta) \qquad (1.1)$$

the condition for continuity of mass flow

$$j = \rho_1 V_1 \sin\sigma = \rho_2 V_2 \sin(\sigma - \vartheta) \qquad (1.2)$$

the condition for conservation of tangential momentum

$$4\pi\rho_1 V[V_2 \cos(\sigma - \vartheta) - V_1 \cos\sigma] = H_1 [H_2' \cos(\sigma - \vartheta) - H_1 \cos\sigma] \qquad (1.3)$$

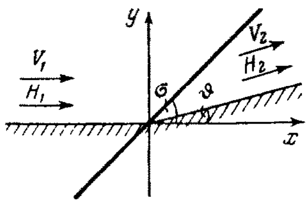

Fig. 1.

the condition for conservation of normal momentum

$$p_2 + \frac{j^2}{\rho_2} + \frac{H_2^2}{8\pi}\cos^2(\sigma - \vartheta) = p_1 + \frac{j^2}{\rho_1} + \frac{H_1^2}{8\pi}\cos^2\sigma \tag{1.4}$$

the condition for conservation of energy

$$\frac{\varkappa}{\varkappa - 1}\frac{p_2}{\rho_2} + \frac{V_2^2}{2} = \frac{\varkappa}{\varkappa - 1}\frac{p_1}{\rho_1} + \frac{V_1^2}{2} \tag{1.5}$$

We have made use of the fact that the angles of inclination θ of the vectors **H** and **V** after the shock wave are identical. This follows from the continuity of the tangential component of the electric field **E**. In an infinitely conducting fluid, $\mathbf{E} = -(1/c)\,\mathbf{V}\times\mathbf{H}$. In our case, **E** is zero ahead of the shock, and thus it follows that also after the shock $\mathbf{V}\times\mathbf{H} = 0$, and so **H** is parallel to **V**.

From (1.1) and (1.2) it follows that

$$\frac{H_2}{\rho_2 V_2} = \frac{H_1}{\rho_1 V_1} \tag{1.6}$$

Let us consider shock waves of small intensity. Let $\mathbf{H}_0$, $\mathbf{V}_0$, p_0 and ρ_0 correspond to the free stream, with $\mathbf{H}_0$ and $\mathbf{V}_0$ in the direction of the x-axis. Let h_x, h_y, v_x, v_y, p and ρ, with indices 1 and 2, be small perturbations, corresponding to upstream and downstream of the shock wave.*

Neglecting the squares of small quantities in (1.1) to (1.5), and rearranging as necessary, we obtain the following relations for a weak shock wave:

$$[h_x]\operatorname{tg}\sigma_0 - [h_y] = 0, \qquad \operatorname{tg}\sigma_0[v_x] - [v_y] + \frac{V_0}{\rho_0}[\rho] = 0$$

$$[p] + \rho_0 V_0[v_x] = 0, \qquad [v_x] + [v_y]\frac{1 - H_0^2/4\pi\rho_0 V_0^2}{M_0^2\sin\sigma_0\cos\sigma_0} = 0 \tag{1.7}$$

* The vectors **V** and **H** ahead of the shock may be non-parallel (but nearly parallel).

** Conditions for the conservation of the tangential component of electric field.

$$H_0 [v_y] + V_0 [h_y] = 0 \ ^2$$

$$\mathrm{tg}^2 \sigma_0 = \frac{1}{M_0^2 - 1} \frac{{}^1/_2 \rho_0^2 V_0^2 - (1 - M_0^2) H_0^2 / 8\pi}{{}^1/_2 \rho_0 V_0^2 - H_0^2 / 8\pi} \qquad (1.8)$$

$$\left(M_0 = \frac{V_0}{a_0}, \ a_0^2 = \frac{\varkappa p_0}{\rho_0}\right)$$

where σ_0 is the angle of inclination of the shock wave of zero intensity, a_0 is the speed of sound of the free stream. The symbol $[A]$ indicates $(A_2 - A_1)$.

The flow under consideration has two characteristic non-dimensional parameters: the Mach number M and the parameter N_0^2, which is equal to the ratio of magnetic and hydrodynamic pressures. Using these parameters we rewrite (1.8) in the form

$$\mathrm{tg}\, \sigma_0 = \pm \sqrt{\frac{M_0^2 - N_0^2(1 - M_0^2)}{(M_0^2 - 1)(M_0^2 - N_0^2)}} \qquad \left(N_0^2 = \frac{H_0^2}{4\pi \varkappa p_0}\right) \qquad (1.9)$$

It is evident that for given N_0 there always exist sufficiently small values of M for which the right hand side of (1.9) is imaginary. For those values of M, weak shock waves do not exist. For $M_0 = M_{01} = \sqrt{N_0^2/(1 + N_0^2)}$ the numerator in the right-hand side of (1.9) changes sign and the right-hand side becomes real.

It follows that for $M_{01} \leqslant M_0 \leqslant \min(1, N_0)$ weak shock waves may exist.

Here, three possibilities present themselves, as depicted in [the hodographs of]* Fig. 2, where the numbers 1, 2, 3 denote, respectively, quasi-hyperbolic and fully hyperbolic regimes:

(a) for $N_0 < 1$, the region of existence of shock waves ends at subsonic velocities, followed, up to $M_0 = 1$, by a region in which there are no shock waves; for $M_0 \geqslant 1$ shock waves again appear (Fig. 2 a);

(b) for $N_0 > 1$, the subsonic region in which shock waves exist extends to $M_0 = 1$, while $M_0 \geqslant 1$ up to $M_0 = N_0$ is a region in which weak shock waves cannot exist; for $M_0 \geqslant N_0$ weak shock waves again exist (Fig. 2 b);

(c) for $N_0 = 1$, both regions of existence of weak discontinuities join at $M = 1$ (Fig. 2 c).

The subsonic region in which weak shock waves exist we shall call quasi-hyperbolic**, and the corresponding supersonic region fully hyperbolic or, simply, hyperbolic.

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* Added in translation.

** In what follows, the reason for this name will become clear.

As is well known [1], the pressure in real*, weak shock waves must increase. Corresponding to (1.7) and (1.8) in the quasi-hyperbolic regime with $\sigma_0 < 1/2\pi$ and $[v_y] > 0$, we must have $[v_x] > 0$, and then the pressure p decreases, i.e. for a real shock wave with $\sigma_0 < 1/2\pi$ the flow has to deflect downward, not upward, as ordinarily in supersonic aerodynamics. In the hyperbolic regime with $\sigma_0 < 1/2\pi$ real shock waves deflect the flow upward.

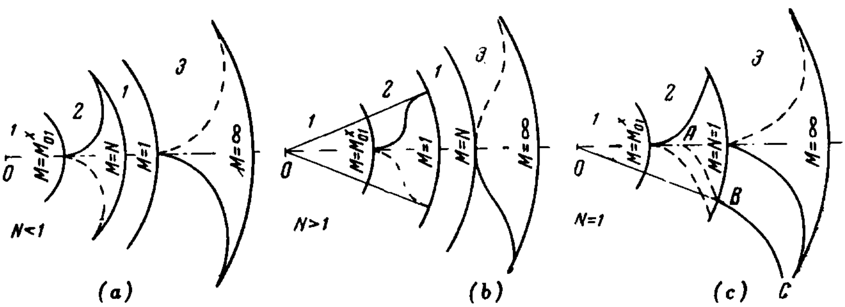

Fig. 2.

Let us denote the angle of inclination of a weak shock wave of non-zero intensity by $\sigma = \sigma_0 + \delta$. Let the flow after the wave (Fig. 1) turn through the angle θ. Expanding relations (1.1)-(1.5) in series of θ and keeping terms of second order, we find

$$\delta = \frac{(3+\varkappa)(1-M_1^2)N_1^2 + (\varkappa+1)(N_1^2 - M_1^2)}{4[M_1^2 - N_1^2(1-M_1^2)](1-M_1^2)} M_1^2\vartheta \qquad (1.10)$$

As shown above, in the quasi-hyperbolic regime a deflection of the flow and the field to positive angles θ occurs for shocks with $\sigma_0 = 1/2\pi$. From (1.10) it follows that $\delta > 0$, and therefore the shock wave approaches the x-axis as θ increases. Shock waves with angles of inclination $|\sigma| < 1/2\pi$ do not exist. In particular, there are no shock waves perpendicular to the flow.

In the hyperbolic regime, positive θ corresponds to $\sigma_0 < 1/2\pi$ and $|\sigma| \geqslant |\sigma_0|$, as in ordinary gas dynamics. For $N > 1$ and $1 \leqslant M \leqslant N$ there evidently exists a normal shock ($\sigma = 1/2\pi$), but there are no weak shocks. Shock waves with angles of inclination $1/2\pi \leqslant \sigma \leqslant \pi$ appear here as strong shock waves.

2. The equations of magnetohydrodynamics and their characteristics. The motion of a gas with infinite conductivity in the presence of a magnetic field is described by the following system of equations [1]:

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* In which entropy increases.

$$\operatorname{div} \rho \mathbf{V} = 0, \qquad (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{\nabla p}{\rho} - \frac{1}{4\pi\rho}\mathbf{H} \times \operatorname{rot} \mathbf{H} \qquad (2.1)$$

$$\operatorname{div} \mathbf{H} = 0, \qquad \operatorname{rot}(\mathbf{V} \times \mathbf{H}) = 0$$

For the case of plane flow it follows from the last equation that $\mathbf{V} \times \mathbf{H}$ = const. Since the flows under consideration are those for which $\mathbf{V}_0 \parallel \mathbf{H}_0$ at infinity, then $\mathbf{V} \parallel \mathbf{H}$ throughout the flow field, since at the shock wave the parallelism of these vectors is not disturbed, as shown in Article 1. Thus, throughout the flow

$$V_x H_y - V_y H_x = 0 \qquad (2.2)$$

Take the x-axis along a streamline. Then $V_y = 0$ and, from (2.2) also $H_y = 0$. We write the continuity equation (2.1) in the form

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + V_x \frac{\partial \ln \rho}{\partial x} = 0 \qquad (2.3)$$

From (2.1) and (2.2) we have

$$\frac{\partial V_y}{\partial y} = \frac{V_x}{H_x}\frac{\partial H_y}{\partial y} = -\frac{V_x}{H_x}\frac{\partial H_x}{\partial x} \qquad (2,4)$$

Eliminating $\partial V_y / \partial y$ from (2.3) we obtain, with (2.4),

$$\frac{\partial}{\partial x} \ln\left(\frac{H_x}{\rho V_x}\right) = 0 \qquad (2.5)$$

Thus, $H/\rho V$ = const along a streamline. This ratio is continuous across a shock wave, according to equation (1.6), thus throughout the flow we have

$$\frac{H}{\rho V} = \frac{H_0}{\rho_0 V_0} \qquad (2.6)$$

In view of the parallelism of the vectors $\mathbf{H}$ and $\mathbf{V}$, equation (2.6) is equivalent to two relations:

$$H_x = \frac{H_0}{\rho_0 V_0}\rho V_x, \qquad H_y = \frac{H_0}{\rho_0 V_0}\rho V_y \qquad (2.7)$$

Using these expressions, we can eliminate the magnetic field from the equations of motion (2.1), and obtain

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{H_0^2 V_y}{4\pi\rho_0^2 V_0^2}\left(\frac{\partial \rho V_x}{\partial y} - \frac{\partial \rho V_y}{\partial x}\right) \qquad (2.8a)$$

$$V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{H_0^2 V_x}{4\pi\rho_0^2 V_0^2}\left(\frac{\partial \rho V_y}{\partial x} - \frac{\partial \rho V_x}{\partial y}\right) \qquad (2.8b)$$

Equations (2.8) together with the continuity equation (2.1) and the condition of constancy of entropy along a streamline,

$$V_x \frac{\partial}{\partial x}\left(\frac{p}{\rho^\varkappa}\right) + V_y \frac{\partial}{\partial y}\left(\frac{p}{\rho^\varkappa}\right) = 0 \tag{2.9}$$

constitute a closed system of equations for the determination of the four unknowns V_x, V_y, p and ρ.

Writing equations (2.8 a) and (2.9) in coordinates whose x-axis is along a streamline we note that the absolute value of the vector V, the pressure p and the density ρ are related, along a streamline, by the same relations as in ordinary gas dynamics:

$$\frac{V^2}{2} + \frac{\varkappa}{\varkappa - 1}\frac{p}{\rho} = \text{const} \tag{2.10}$$

or

$$p = \left[\frac{\varkappa - 1}{2\varkappa}(V_{\max}{}^2 - V^2)\right]^{\frac{\varkappa}{\varkappa-1}} f^{-\frac{1}{\varkappa-1}} \qquad \left(f = \frac{p}{\rho^\varkappa}\right) \tag{2.11}$$

Here $V_{\max}$, the maximum flow speed, is a constant for the flow, as follows from (1.5).

The change of entropy in shocks is proportional to the cube of the pressure jump [1]. Therefore, to second order, f = const, and thus p has a one;to-one relation with V, as in ordinary gas dynamics.

Introducing the usual procedures for finding the characteristics of the system of equations, we find that plane flows of magnetogasdynamics possess two systems of characteristics*, whose angles of inclination σ_0 to the streamline are determined by the expressions

$$\operatorname{tg}\sigma_0 = \pm\sqrt{\frac{M^2 - N^2(1 - M^2)}{(M^2 - 1)(M^2 - N^2)}} \qquad \left(N = \frac{H^2}{8\pi} \Big/ \frac{\varkappa p}{2}\right) \tag{2.12}$$

The characteristics corresponding to the upper sign we shall call characteristics of the first family, those of the lower sign, the second family. Since (2.12) is the same as (1.9), the characteristics are real in those ranges of M and N in which weak shock waves exist. As required, the characteristics coincide with the shock waves of vanishing intensity. Along these characteristics, the required functions are related by the relations:

$$\mp(M^2 - N^2)\,|\operatorname{tg}\sigma_0|\,d\vartheta - N^2 d\ln H - \frac{d\ln p}{\varkappa} = 0 \tag{2.13}$$

* The streamline also appears as a fourth-order characteristic, along which the four relations (2.7), (2.9), (2.11) hold.

where the upper and lower signs correspond with those in (2.12). If the entropy is constant throughout the flow field, then (2.13) may be re-written in the following form:

$$\pm(M^2 - N^2)\,|\operatorname{tg}\sigma_0|\,d\vartheta + [M^2 - N^2(1 - M^2)]\,d\ln V = 0 \qquad (2.14)$$

From (2.13) and (2.14) it may be seen that, along one and the same family of characteristics in the hyperbolic ($M > N$) and quasi-hyperbolic ($M < N$) regions, the directions of changing velocity and pressure, for the same change of θ, are opposite. In the hodograph plane the characteristics have the form shown in Fig. 2.*

In the hyperbolic regime, for $N \to 0$ all the formulas reduce to the corresponding formulas of ordinary gas dynamics.

3. Simple waves. As in ordinary gas dynamics, there exist in each hyperbolic regime two types of simple waves: condensation and rarefaction waves.

Every wave maps into a characteristic in the hodograph plane. In magnetogasdynamics, the properties of simple waves are different in each hyperbolic regime.

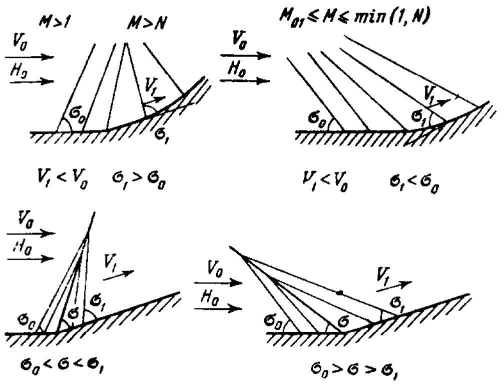

Fig. 3.

Corresponding to equation (2.14) and Fig. 2, the flows along concave and convex walls have the form shown in Figs. 3 and 4. The behavior of the flows near the boundary lines ($M = M_{01}$, $M = N$, etc.) is essentially different for $N > 1$ and $N < 1$. For $N < 1$, in the vicinity of the lines $M = M_{01}$ and $M = 1$, small changes of angle correspond to large changes of

* In Fig. 2, characteristics of the first family are shown by dotted lines, those of the second family by continuous lines.

speed. At the outer boundaries of the hyperbolic regions ($M = N$ and $M \to \infty$), a very large curvature of the streamline is required to change the speed. For $N > 1$, on both boundaries of the quasi-hyperbolic region, the magnitude of the velocity vector changes suddenly for a small turn. On the other hand, on both boundaries of the fully hyperbolic region, a change in the magnitude of the velocity vector requires a large turn.

Of interest is the case $N = 1$ (Fig. 5). In this case an expansion for example, proceeds up to $M = 1$ along a characteristic of the first family AB (Fig. 2c), and for $M > 1$ along a characteristic of the second family BC. The inclination of characteristics in the physical plane changes continuously, going through $1/2\pi$ at $M = 1$.

4. Flow around finite bodies. Assume a body in a flow of gas of infinite conductivity having the velocity $\mathbf{V}_0$ at infinity, in a magnetic field $\mathbf{H}_0$, parallel to $\mathbf{V}_0$. We will consider a body thin enough so that if the approaching stream belongs, for example, to one of the hyperbolic flow regimes, then all portions of the flow are within the boundaries of that regime.

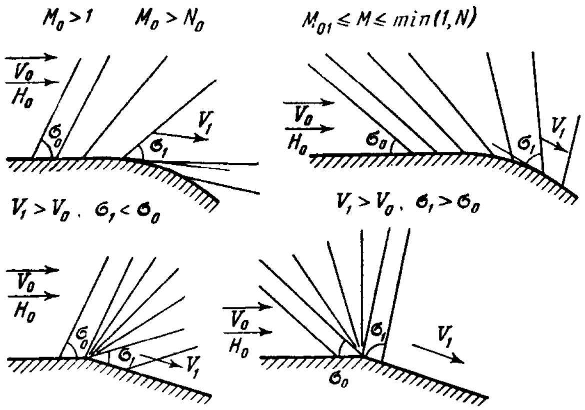

Fig. 4.

In the fully hyperbolic regime the flow is analogous to supersonic flow over a profile (Fig. 6).

If cubes of the velocity perturbations are neglected, then, as shown above, the flow is isentropic. To this order, changes in a shock wave are the same as changes in a simple wave. Therefore the flow around a profile can be constructed by Busemann's method. In the quasi-hyperbolic region an analogous solution can be formally constructed (following the scheme of Fig. 6), in which condensations are everywhere replaced by rarefactions. However, such a flow cannot exist, since in that case the entropy would have to decrease in a shock wave. The impossibility of such a flow is also evident from other considerations. In the flow which is constructed according to Fig. 6, the flow upstream of the first shock is undisturbed. It is known [1] that in magnetohydrodynamics there exist two types of wave, propagating in all directions with two different speeds. The speed of propagation u of these waves is determined by the equation [1]

$$(u^2 - a_0^2)(u^2 - N^2 a_0^2 \sin^2 \sigma) = u^2 N^2 \cos^2 \sigma \qquad (4.1)$$

where a_0 is the speed of sound in the basic flow and σ is the angle of inclination of the wave front to the x-axis, along which the vectors $\mathbf{V}_0$ and $\mathbf{H}_0$ are directed, in our case. Waves with speeds a_0 and Na_0 evidently propagate against the flow ($\sigma = 1/2\pi$). Since in the quasi-hyperbolic

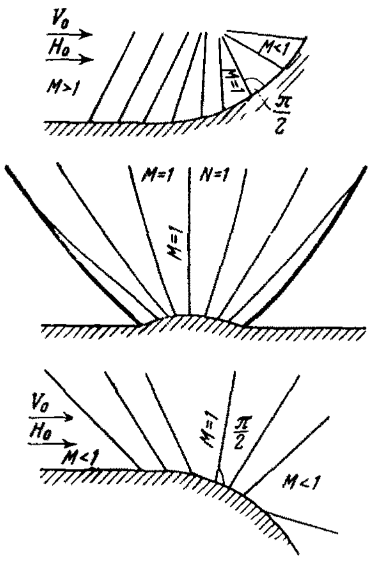

Fig. 5.

region $M < 1$ and $M < N$, both waves penetrate upstream. As the inclination of the wave front decreases, the speed of one of the waves decreases to zero as $\sigma \to 0$, while that of the other increases to $(N_0 + 1)a_0$. For $N < 1$, there is a decrease in the velocity of that family of waves whose speed

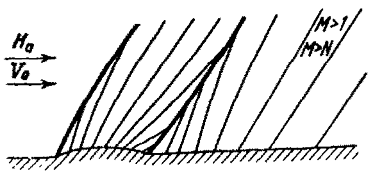

Fig. 6.

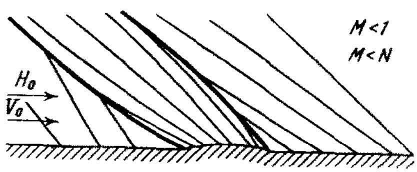

Fig. 7.

is Na_0 at $\sigma = 1/2\pi$. For $N > 1$, on the other hand, the speed of propagation of this family of waves increases as $\sigma \to 0$, while the speed of the second family (which has speed a_0 at $\sigma = 1/2\pi$) decreases. Thus, one of the families always penetrates upstream. The second family, beginning at some $\sigma < 1/2\pi$, cannot penetrate upstream. That such a family must exist is shown by the appearance of characteristics for $M < 1$. Evidently, the angle σ_0 is the angle of inclination of the characteristics.

Thus, in the quasi-hyperbolic regime, disturbances can penetrate upstream. On the other hand, if characteristics of the second family could come from the undisturbed flow at upstream infinity, then the uniquely possible solution would be the solution constructed according to the scheme of Fig. 6, with decreasing entropy. Therefore the characteristics have to run into a shock wave which runs upstream, as shown in Fig. 7. In this case the characteristics running into the undisturbed flow downstream of the body are of the first family. The whole flow is obtained as by a mirror reflection of the usual supersonic flow of ordinary gas dynamics. However, there is an important difference between these two flows. In supersonic aerodynamics, as well as in fully hyperbolic flows of magnetogasdynamics, the flow ahead of the shock is undisturbed. Thus the flow ahead of the bow shock is known, and it is possible, with the method of characteristics, to construct the whole flow and shock waves, step by step, beginning at the nose of the body. In analogy with that, it is natural, in the quasi-hyperbolic region, to try to calculate the flow by the method of characteristics, going from the end of the body upstream. However, in this case the flow downstream of the trailing-edge shock wave is disturbed, due to the change of entropy in the shocks. Therefore, in the general case, all parts of the flow are interdependent, and the usual method of characteristics does not give a method of constructing the flow. In order to emphasize this peculiarity which is characteristic of elliptical regions, we have called these flows quasi-hyperbolic.

If we restrict ourselves to the second-order approximation and neglect the changes of entropy, then the indicated property of interdependence disappears and the solution can be constructed entirely by analogy with the Busemann method. The only difference in the given case is that we go along characteristics of the first family, from downstream infinity, rather than along characteristics of the second family, from upstream infinity. Correspondingly, in the hodograph plane, the whole flow maps into a characteristic of the first family, not the second.

In the case under consideration, the magnetic field **H** is everywhere parallel to the velocity vector **V**, as shown above. Therefore, on the boundary of the body the normal component of the magnetic field is equal to zero. Inasmuch as there are no magnetic sources inside the body, then in accordance with Maxwell's equations the field must be zero inside the body. Thus on the boundary of the body there is a tangential discontinuity in the magnetic field and corresponding to it a surface current. The magnetic field tries to squeeze this current to the body, which in fact determines the magnetic pressure on the body, $H^2/8\pi$. This effect is entirely analogous to the pinch-effect.

5. Linearized theory. Let the free stream velocity V_0 be along the x-axis, and let the field again by H_0 at infinity, parallel to the velocity. Let v_x, u_y, h_x, h_y, p and ρ denote the perturbations of velocity, field, etc. Neglecting squares of these quantities, from (2.1) and (2.8) we obtain

$$(1 - M_0^2)\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{5.1}$$

$$[M_0^2 - N_0^2(1 - M_0^2)]\frac{\partial v_x}{\partial y} - [M_0^2 - N_0^2]\frac{\partial v_y}{\partial x} = 0$$

The inclination of the characteristics of this system of equations is evidently determined by equation (1.9). Along the characteristics, relations (2.14) are satisfied; after linearization these have the form

$$\pm (M_0^2 - N_0^2)\,|\operatorname{tg}\sigma_0|\,dv_y + [M_0^2 - N_0^2(1 - M_0^2)]\,dv_x = 0 \tag{5.2}$$

The full pressure coefficient is

$$\bar{p}_n = \frac{p + p_m}{{}^1/_2\rho_0 V_0^2} = -2\,\frac{M_0^2 - N_0^2(1 - M_0^2)}{M_0^2}\,\frac{v_x}{V_0} \tag{5.3}$$

where $p = -\rho_0 V_0 v_x$ is the perturbation hydrodynamic pressure and $p_m = 1/4\, H_0 h_x = H_0^2\,(1 - M_0^2)v_x/4\pi V_0$ is the magnetic pressure.

Let us bring (5.1) into a form which is conventional in the linearized theory of compressible fluid flow. For this, we introduce the velocity

$$v_x^* = \frac{M_0^2 - N_0^2(1 - M_0^2)}{M_0^2 - N_0^2}\,v_x \tag{5.4}$$

Then instead of (5.1) we obtain

$$\frac{(1 - M_0^2)(M_0^2 - N_0^2)}{M_0^2 - N_0^2(1 - M_0^2)}\,\frac{\partial v_x^*}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \qquad \frac{\partial v_x^*}{\partial y} - \frac{\partial v_y}{\partial x} = 0 \tag{5.5}$$

The coefficient of $\partial v_x^*/\partial x$ is evidently positive in elliptic regions and negative in hyperbolic regions. Correspondingly, the system (5.5) reduces either to Laplace's equation or to the wave equation.

In accordance with (5.4)-(5.5), in the first elliptic, subsonic region ($M < M_{01}$) the character of the propagation of velocity will be the same as in an ideal incompressible fluid. However, according to (5.3), the regions of rarefaction are replaced by regions of condensation. The region which has all the properties of subsonic flows turns out to be the second subsonic elliptic zone, which exists for $N \leqslant M \leqslant 1$. For $N \to 0$ this region spreads and encompasses the whole subsonic regime. According to (5.4)-(5.5) the picture of the velocities in the elliptic supersonic region, and according to (5.3) the picture of the pressures also, is opposite to the picture to which we are accustomed in ideal incompressible flow. Here, both the velocity perturbations and the pressure perturbations have the opposite sign.

In the hyperbolic regions (Fig. 8), going from infinity along characteristics of the second family for $M > 1$ and along characteristics of the first family for $M < 1$, we obtain, corresponding to (5.2), the expression

$$\frac{v_x}{V_0} = -\vartheta \sqrt{\frac{M_0^2 - N_0^2}{(M_0^2 - 1)[M_0^2 - N_0^2(1 - M_0^2)]}} \tag{5.6}$$

It follows that for this case

$$\bar{p}_n = \frac{2\vartheta}{M_0^2} \sqrt{\frac{(M_0^2 - N_0^2)[M_0^2 - N_0^2(1 - M_0^2)]}{M_0^2 - 1}} \tag{5.7}$$

This expression differs only by a factor from the usual expression $p = 2\theta / \sqrt{M_0^2 - 1}$, of supersonic aerodynamics.

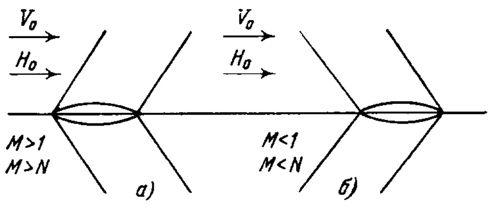

Fig. 8.

From here it follows that in the case under consideration with $M_0 < 1$ there exist flows (for $M_{01} \leqslant M_0 \leqslant \min(N_0, 1)$), for which D'Alembert's paradox is not fulfilled. In the elliptic regime, on the other hand, the paradox does occur, at least within the linearized theory*. Therefore, for $M_0 = 1$ it is always possible to choose a magnetic field H_0 such that the drag of the body becomes zero. On the other hand, for $M < 1$ it is possible to choose a field such that a wave drag appears on the body. These possibilities do not exist in ordinary aerodynamics.

* With finite disturbances in the elliptic zones, strong shock waves may appear (cf. Art. 1).

6. Three-dimensional flow. The plane flows discussed above are not plane in the strict sense of the word, since the currents which appear are perpendicular to the plane of the flow. If in any part of a three-dimensional flow (in which currents are closed) the variations of the flow and field parameters in the z-direction are small compared to variations in the xy-plane, then the flow in that plane is considered to be plane. The flow at a shock wave is by its very nature plane [1], since the vectors **V** and **H** before and after the shock and the normal to the wave **n** lie in one plane. Therefore the relations in the shock waves are the same as those given above, if the angles σ and θ are measured in the plane passing through the vectors **V**, **H** and **n**.

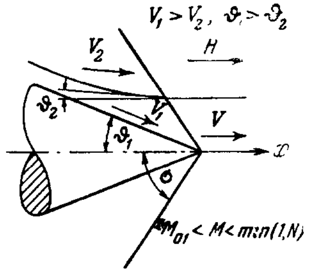

Fig. 9.

If the vectors **V** and **H** are parallel ahead of a three-dimensional shock wave, then they are parallel after it. Using this property and (2.4) we find that **V**$\|$**H** in the whole three-dimensional flow if these vectors are parallel at infinity. It is easy to show, by the usual methods, that for those relationships between **V** and **H** for which characteristics exist in plane flow, there exist in the three-dimensional case characteristic surfaces and cones whose half-angle is determined by expression (2.12).

Let us consider the flow around a body of revolution, with the vectors $\mathbf{V}_0$ and $\mathbf{H}_0$ at infinity parallel to its axis. The character of the flow around the body in each of the flow regimes considered in article 4 will be the same as in the plane case.

A peculiarity appears in the quasi-hyperbolic region. Here, just as in the plane case, the flow has the character shown in Fig. 7.

In ordinary supersonic aerodynamics, one of the few exact solutions is for conical flow over a circular cone with an attached bow wave. In the case under consideration an analogous flow is obtained for an infinite reverse (trailing edge) cone (Fig. 9). However, here there is a decreased pressure on the cone surface, and the pressure increases as one goes away from the cone toward the shock.

In supersonic aerodynamics, if a finite body has a conical nose, then the flow around that nose is the same as over an infinite cone. In the quasi-hyperbolic case the flow around a conical tail will not be conical in general, due to the changes of entropy in the bow and tail shock waves. If the changes of entropy are neglected (i.e. in the linearized and second-order approximations) then the computation of the flow is analogous to that for ordinary supersonic flow around an axisymmetric body. But here the computation begins from the tail end of the body, as though there were a reverse flow over the body.

Let us consider the flow around a body of revolution in the linearized approximation. In cylindrical coordinates the equations of motion have the form:

$$(1-M_0^2)\frac{\partial v_x}{\partial x}+\frac{\partial v_r}{\partial r}+\frac{v_r}{r}=0$$

$$[M_0^2-N_0^2(1-M_0^2)]\frac{\partial v_x}{\partial r}+(N_0^2-M_0^2)\frac{\partial v_r}{\partial x}=0 \tag{6.1}$$

where v_x and v_r are velocity perturbations in the x- and r-directions. Making the substitutions

$$x=-x_1,\quad r=r_1,\quad v_x=\frac{N_0^2-M_0^2}{M_0^2-N_0^2(1-M_0^2)}v_x^*,\quad v_r=v_r^*$$

we obtain

$$\beta^2\frac{\partial v_x^*}{\partial x_1}-\frac{\partial v_r^*}{\partial r_1}-\frac{v_r^*}{r_1}=0,\qquad \frac{\partial v_x^*}{\partial r_1}-\frac{\partial v_r^*}{\partial x_1}=0,\ \text{if}\ \beta^2=\frac{(1-M_0^2)(N_0^2-M_0^2)}{M_0^2-N_0^2(1-M_0^2)} \tag{6.2}$$

This system is identical with the corresponding system of equations of supersonic aerodynamics. The solution of this system for a slender body has the form

$$v_x^*=\frac{V_0}{2\pi}\int_0^{x_1-\beta r_1}\frac{d^2S}{dx_1^2}\frac{d\xi_1}{\sqrt{(x_1-\xi_1)^2-\beta^2r_1^2}},\qquad v_r^*=-\frac{V_0}{2\pi r_1}\int_0^{x_1-\beta r_1}\frac{d^2S}{dx_1^2}\frac{(x_1-\xi_1)\,d\xi_1}{\sqrt{(x_1-\xi_1)^2-\beta^2r_1^2}} \tag{6.3}$$

where S is the cross-sectional area of the body, and the integration proceeds along x_1 (i.e. from the tail toward the nose of the body). As might have been expected, there is a rarefaction on the conical tail portion of the body.

It is known that in supersonic flow of a non-conducting gas there is a reduction of pressure on the tail portion of a body of revolution which is pointed at both ends. At the very tail there appears a subsonic zone, and the tail wave actually begins at some distance from the tail. In the quasi-hyperbolic case an analogous flow occurs at the nose portion of the body.

Using the analogy with ordinary aerodynamics, it would be possible to construct a number of exact solutions and indicate the special properties of those or other flows. However, we have attempted here to indicate mainly those properties of magnetohydrodynamic flows which do not occur in ordinary aerodynamics.

In the present paper we have not considered the important class of internal flows in plane and axisymmetric channels (nozzles, diffusers, etc.) in a longitudinal magnetic field. To this question, and to the analysis of a more general class of flows with non-parallel velocity vector and magnetic field, we shall devote separate investigations.

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